

**Review.** Recall the Gauss–Jordan method of computing  $A^{-1}$ . Starting with the augmented matrix  $[A \mid I]$ , we do Gaussian elimination until we obtain the RREF, which will be of the form  $[I \mid A^{-1}]$  so that we can read off  $A^{-1}$ .

**Why does that work?** By our discussion, the steps of Gaussian elimination can be expressed by multiplication (on the left) with a matrix  $B$ . Only looking at the first part of the augmented matrix, and since the RREF of an invertible matrix is  $I$ , we have  $BA = I$ , which means that we must have  $B = A^{-1}$ . The other part of the augmented matrix (which is  $I$  initially) gets multiplied with  $B = A^{-1}$  as well, so that, in the end, it is  $BI = A^{-1}$ . That's why we can read off  $A^{-1}$ !

**For instance.** To invert  $\begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix}$  using the Gauss–Jordan method, we would proceed as follows:

$$\left[ \begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 4 & -6 & 0 & 1 \end{array} \right] \xrightarrow{R_2 - 2R_1 \Rightarrow R_2} \left[ \begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & -8 & -2 & 1 \end{array} \right] \xrightarrow{-\frac{1}{8}R_2 \Rightarrow R_2} \left[ \begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{4} & -\frac{1}{8} \end{array} \right] \xrightarrow{R_1 - \frac{1}{2}R_2 \Rightarrow R_1} \left[ \begin{array}{cc|cc} 1 & 0 & \frac{3}{8} & \frac{1}{16} \\ 0 & 1 & \frac{1}{4} & -\frac{1}{8} \end{array} \right]$$

We conclude that  $\begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{3}{8} & \frac{1}{16} \\ \frac{1}{4} & -\frac{1}{8} \end{bmatrix}$ .

Of course, for  $2 \times 2$  matrices it is much simpler to use the formula  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ .

## Review: Vector spaces, bases, dimension, null spaces

### Review.

- Vectors are things that can be **added** and **scaled**.
- Hence, given vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$ , the most general we can do is form the **linear combination**  $\lambda_1 \mathbf{v}_1 + \dots + \lambda_n \mathbf{v}_n$ . The set of all these linear combinations is the **span** of  $\mathbf{v}_1, \dots, \mathbf{v}_n$ , denoted by  $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ .
- Vector **spaces** are spans.

**Equivalently.** Vector spaces are sets of vectors so that the result of adding and scaling remains within that set.

**Homework.** Of course, the latter is a very informal statement. Revisit the formal definition, probably consisting of a list of axioms, and observe how that matches with the above (for instance, several of the axioms are concerned with addition and scaling satisfying the “expected” rules).

- Recall that vectors from a vector space  $V$  form a **basis** of  $V$  if and only if
  - the vectors span  $V$ , and
  - the vectors are (linearly) independent.

**Equivalently.**  $\mathbf{v}_1, \dots, \mathbf{v}_n$  from  $V$  form a basis of  $V$  if and only if every vector in  $V$  can be expressed as a unique linear combination of  $\mathbf{v}_1, \dots, \mathbf{v}_n$ .

**Just checking.** Make sure that you can define precisely what it means for vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$  to be independent.

- The **dimension** of a vector space  $V$  is the number of vectors in a basis for  $V$ .

No matter what basis one chooses for  $V$ , it always has the same number of vectors.

**Example 12.**  $\mathbb{R}^3$  is the vector space of all vectors with 3 real entries.

$\mathbb{R}$  itself refers to the set of real numbers. We will later also discuss  $\mathbb{C}$ , the set of complex numbers.

The **standard basis** of  $\mathbb{R}^3$  is  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ . The dimension of  $\mathbb{R}^3$  is 3.

**Review.** The **null space**  $\text{null}(A)$  of a matrix  $A$  consists of those vectors  $\mathbf{x}$  such that  $A\mathbf{x} = \mathbf{0}$ .

Make sure that you see why  $\text{null}(A)$  is a vector space. [For instance, if you pick two vectors in  $\text{null}(A)$  why is it that the sum of them is in  $\text{null}(A)$  again?]

**Example 13.** What is  $\text{null}(A)$  if the matrix  $A$  is invertible?

**Solution.** If  $A$  is invertible, then  $A\mathbf{x} = \mathbf{0}$  has the unique solution  $\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}$ .

Hence,  $\text{null}(A) = \{\mathbf{0}\}$  which is the trivial vector space (consisting of only the null vector) and has dimension 0.

**Example 14.** Compute a basis for  $\text{null}(A)$  where  $A = \begin{bmatrix} -1 & 0 & 2 \\ 2 & -3 & 2 \\ 1 & 0 & -2 \end{bmatrix}$ .

**Solution.** We perform row operations and obtain

$$\text{null}\left(\begin{bmatrix} -1 & 0 & 2 \\ 2 & -3 & 2 \\ 1 & 0 & -2 \end{bmatrix}\right) \xrightarrow[R_3+R_1 \Rightarrow R_3]{R_2+2R_1 \Rightarrow R_2} \text{null}\left(\begin{bmatrix} -1 & 0 & 2 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix}\right) \xrightarrow[-\frac{1}{3}R_2 \Rightarrow R_2]{-R_1 \Rightarrow R_1} \text{null}\left(\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\right).$$

From the RREF, we can now read off the general solution to  $A\mathbf{x} = \mathbf{0}$ :

- $x_1$  and  $x_2$  are pivot variables. [For each we have an equation expressing it in terms of the other variables; for instance,  $x_1 - 2x_3 = 0$  tells us that  $x_1 = 2x_3$ .]
- $x_3$  is a free variable. [There is no equation forcing a value on  $x_3$ .]
- Hence, without computation, we see that the general solution is  $\begin{bmatrix} 2x_3 \\ 2x_3 \\ x_3 \end{bmatrix}$ .

In other words, a basis is  $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ .

**Comment.** We are starting with the three equations  $-x_1 + 2x_3 = 0$ ,  $2x_1 - 3x_2 + 2x_3 = 0$ ,  $x_1 - 2x_3 = 0$ . Performing row operations on the matrix is the same as combining these equations (with the objective to form simpler equations by eliminating variables).

**Example 15.** Compute a basis for  $\text{null}\left(\begin{bmatrix} 2 & 0 & 2 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}\right)$ .

**Solution.**

$$\text{null}\left(\begin{bmatrix} 2 & 0 & 2 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}\right) \xrightarrow[R_3 - \frac{1}{2}R_1 \Rightarrow R_3]{R_2 - R_1 \Rightarrow R_2} \text{null}\left(\begin{bmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \xrightarrow{\frac{1}{2}R_1 \Rightarrow R_1} \text{null}\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

This time,  $x_2$  and  $x_3$  are free variables. The general solution is  $\begin{bmatrix} -x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ .

Hence, a basis is  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ .