

## Review: Matrix calculus

**Example 1.** Matrix multiplication is not commutative!

- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 3 & 10 \end{bmatrix}$

Multiplication (on the right) with that “almost identity matrix” is performing the column operation  $C_2 + 2C_1 \Rightarrow C_2$  (i.e. 2 times the first column is added to the second column).

- $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 3 & 4 \end{bmatrix}$

Multiplication (on the left) with the same matrix is performing the row operation  $R_1 + 2R_2 \Rightarrow R_1$ .

**First comment.** This indicates a second interpretation of matrix multiplication: instead of taking linear combinations of columns of the first matrix, we can also take linear combinations of rows of the second matrix.

**Second comment.** The row operations we are doing during Gaussian elimination can be realized by multiplying (on the left) with “almost identity matrices”.

**Example 2.**  $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = [14]$  whereas  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [1 \ 2 \ 3] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ .

If you know about the dot product, do you see a connection with the first case?

**Example 3.** Suppose  $A$  is  $m \times n$  and  $B$  is  $p \times q$ . When does  $AB$  make sense? In that case, what are the dimensions of  $AB$ ?

$AB$  makes sense if  $n = p$ . In that case,  $AB$  is a  $m \times q$  matrix.

**Example 4.**  $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

On the RHS we have the **identity matrix**, usually denoted  $I$  or  $I_2$  (since it's the  $2 \times 2$  identity matrix here).

Hence, the two matrices on the left are inverses of each other:  $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$ ,  $\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ .

**Example 5.** The following formula immediately gives us the inverse of a  $2 \times 2$  matrix (if it exists). It is worth remembering!

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{provided that } ad - bc \neq 0$$

Let's check that!  $\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} ad - bc & 0 \\ -cb + ad & ad - bc \end{bmatrix} = I_2$

In particular, a  $2 \times 2$  matrix  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  is invertible  $\iff ad - bc \neq 0$ .

Recall that this is the **determinant**:  $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$ .

In particular:

$$\det(A) = 0 \iff A \text{ is not invertible}$$

Similarly, for  $n \times n$  matrices  $A$ :

$A$ is invertible	(i.e. there is a matrix $A^{-1}$ such that $AA^{-1} = I$ )
$\Leftrightarrow \det(A) \neq 0$	
$\Leftrightarrow Ax = b$ has a unique solution	(namely, $x = A^{-1}b$ )

**Comment.** Why is it not common to write  $\frac{1}{A}$  instead of  $A^{-1}$ ?

The notation  $\frac{1}{A}$  easily leads to ambiguities: for instance, should  $\frac{B}{A}$  mean  $BA^{-1}$  or should it mean  $A^{-1}B$ ?

[Of course, one could try to avoid this by notations like  $B/A$  which would more clearly mean  $BA^{-1}$ . It's just not common and doesn't have any real advantages.]

### Example 6.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -7 & 2 & 3 \\ -16 & 5 & 6 \\ -25 & 8 & 9 \end{bmatrix}$$

Multiplication (on the right) with that "almost identity matrix" is performing the column operation  $C_1 - 4C_2 \Rightarrow C_1$  (i.e.  $-4$  times the second column is added to the first column).

$$\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{bmatrix}$$

Multiplication (on the left) with the same matrix is performing the row operation  $R_2 - 4R_1 \Rightarrow R_2$ .

**Comment (again).** The row operations we are doing during Gaussian elimination can all be realized by multiplying (on the left) with "almost identity matrices".

These matrices are called **elementary matrices** (they are obtained by performing a single elementary row operation on an identity matrix).

Elementary matrices are **invertible** because elementary row operations are reversible:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & \\ & \frac{1}{2} & \\ & & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$