Example 82. (warmup) Consider $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

- What are the eigenspaces?
- What are A^{-1} and A^{100}?

Solution.

- $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a 2-eigenvector, and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a 3-eigenvector. In other words, the 2-eigenspace is $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ and the 3-eigenspace is $\operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$.
- $A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$ and $A^{100} = \begin{bmatrix} 2^{100} & 0 \\ 0 & 3^{100} \end{bmatrix}$

Comment. Algebraically, this looks like a very simple map. However, notice that it is not so easy to say what happens to, say, $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ geometrically. That is because two things are happening: part of that vector is scaled by 2, the other part is scaled by 3.

Example 83. If A has λ-eigenvector $\boldsymbol{v}$, then what can we say about A^2?

Solution. A^2 has λ^2-eigenvector $\boldsymbol{v}$.

[Indeed, $A^2\boldsymbol{v} = A(A\boldsymbol{v}) = A(\lambda\boldsymbol{v}) = \lambda A\boldsymbol{v} = \lambda^2\boldsymbol{v}$. This is even easier in words: multiplying $\boldsymbol{v}$ with A has the effect of scaling it by λ; hence, multiplying it with A^2 scales it by λ^2.]

Important comment. Similarly, A^{100} has λ^{100}-eigenvector $\boldsymbol{v}$.

Example 84. If a matrix A can be diagonalized as $A = PDP^{-1}$, what can we say about A^n?

Solution. First, note that $A^2 = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1}$. Likewise, $A^n = PD^nP^{-1}$.

The point being that D^n is trivial to compute because D is diagonal.

In particular. $A^{-1} = PD^{-1}P^{-1}$

Theorem 85. (spectral theorem, compact version) A symmetric matrix A can always be diagonalized as $A = PDP^T$, where P is orthogonal and D is diagonal (and both are real).

How? We proceed as in the diagonalization $A = PDP^{-1}$. We then arrange P to be orthogonal, by normalizing its columns. If there is a repeated eigenvalue, then we also need to make sure to pick an orthonormal basis for the corresponding eigenspace (for instance, using Gram–Schmidt).

Advanced comment. A matrix such that $A^TA = AA^T$ is called **normal**. In a similar spirit as in Example 87 one can show that, for normal matrices, the eigenspaces are orthogonal to each other. However, normal matrices which are not symmetric will always have complex eigenvalues. (In that case, the orthogonal matrix P gets replaced with a unitary matrix, the complex version of orthogonal matrices, and the P^T becomes the conjugate transpose $P^* = \bar{P}^T$.)

Example 86. (again) Diagonalize the symmetric matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ as $A = PDP^T$.

Solution. See Example 81 for a solution that illustrates how to diagonalize any symmetric matrix.

Here, let us observe that (because the row sums are equal!) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a 4-eigenvector.

Because the eigenspaces are orthogonal (since A is symmetric!), $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ must be an eigenvector. $\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$, so the corresponding eigenvalues is -2.

We normalize the two eigenvectors and use them as the columns of P, so that $P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ is an orthogonal matrix ($P^{-1} = P^T$). With $D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$ we then have $A = PDP^T$.