Example 11. (review) $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \end{bmatrix}$ whereas $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$.

Review: Examples of differential equations we can solve

Let's start with one of the simplest (and most fundamental) differential equations (DE). It is **first-order** (only a first derivative) and **linear** with constant coefficients.

Example 12. Solve $y' = 3y$.

Solution. $y(x) = Ce^{3x}$

Check. Indeed, if $y(x) = Ce^{3x}$, then $y'(x) = 3Ce^{3x} = 3y(x)$.

Comment. Recall we can always easily check whether a function solves a differential equation. This means that (although you might be unfamiliar with the techniques for solving) you can use computer algebra systems like Sage to solve differential equations without trust issues.

To describe a unique solution, additional constraints need to be imposed.

Example 13. Solve the **initial value problem** (IVP) $y' = 3y$, $y(0) = 5$.

Solution. This has the unique solution $y(x) = 5e^{3x}$.

The following is a **nonlinear** differential equation. In general, such equations are much more complicated than linear ones. We can solve this particular one because it is **separable**.

Example 14. Solve $y' = xy^2$.

Solution. This DE is separable: $\frac{1}{y^2}\mathrm{d}y = x\,\mathrm{d}x$. Integrating, we find $-\frac{1}{y} = \frac{1}{2}x^2 + C$.

Hence, $y = -\frac{1}{\frac{1}{2}x^2 + C} = \frac{2}{D - x^2}$.

[Here, $D = -2C$ but that relationship doesn't matter; it only matters that the solution has a free parameter.]

Note. Note that we did not find the solution $y = 0$ (lost when dividing by y^2). It is called a **singular solution** because it is not part of the **general solution** (the one-parameter family found above). [Although, we can obtain it from the general solution by letting $D \to \infty$.]

Check. Compute y' and verify that the DE is indeed satisfied.

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Review: Linear DEs

Linear DEs of order n are those that can be written in the form

$$y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \ldots + P_1(x) y' + P_0(x) y = f(x).$$

The corresponding **homogeneous linear DE** is the DE

$$y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \ldots + P_1(x) y' + P_0(x) y = 0,$$

and it plays an important role in solving the original linear DE.

Important. Note that a linear DE is **homogeneous** if and only if the zero function $y(x) = 0$ is a solution.

In terms of $D = \frac{\mathrm{d}}{\mathrm{d}x}$, the original DE becomes: $Ly = f(x)$ where L is the **differential operator**

$$L = D^n + P_{n-1}(x) D^{n-1} + \ldots + P_1(x) D + P_0(x).$$

The corresponding homogeneous linear DE is $Ly = 0$.

Linear DEs have a lot of structure that makes it possible to understand them more deeply. Most notably, their general solution always has the following structure:

(general solution of linear DEs) For a linear DE $Ly = f(x)$ of order n, the general solution always takes the form

$$y(x) = y_p(x) + C_1 y_1(x) + \ldots + C_n\, y_n(x),$$

where y_p is any single solution (called a **particular solution**) and $y_1, y_2, \ldots, y_n$ are solutions to the corresponding **homogeneous** linear DE $Ly = 0$.

Comment. If the linear DE is already homogeneous, then the zero function $y(x) = 0$ is a solution and we can use $y_p = 0$. In that case, the general solution is of the form $y(x) = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n$.

Why? The key to this is that the differential operator L is **linear**, meaning that, for any functions $f_1(x), f_2(x)$ and any constants c_1, c_2, we have

$$L(c_1 f_1(x) + c_2 f_2(x)) = c_1 L(f_1(x)) + c_2 L(f_2(x)).$$

If this is not clear, consider first a case like $L = D^n$ or work through the next example for the order 2 case.

Example 15. (extra) Suppose that $L = D^2 + P(x) D + Q(x)$. Verify that the operator L is linear.

Solution. We need to show that the operator L satisfies

$$L(c_1 f_1(x) + c_2 f_2(x)) = c_1 L(f_1(x)) + c_2 L(f_2(x))$$

for any functions $f_1(x), f_2(x)$ and any constants c_1, c_2. Indeed:

$$\begin{aligned} L(c_1 f_1 + c_2 f_2) &= (c_1 f_1 + c_2 f_2)'' + P(x)(c_1 f_1 + c_2 f_2)' + Q(x)(c_1 f_1 + c_2 f_2) \\ &= c_1 \{ f_1'' + P(x) f_1' + Q(x) f_1 \} + c_2 \{ f_2'' + P(x) f_2' + Q(x) f_2 \} \\ &= c_1 \cdot L f_1 + c_2 \cdot L f_2 \end{aligned}$$

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Example 16. (extra) Consider the following DEs. If linear, write them in operator form as $Ly = f(x)$.

(a) $y'' = xy$

(b) $x^2 y'' + xy' = (x^2+4)y + x(x^2+3)$

(c) $y'' = y' + 2y + 2(1-x-x^2)$

(d) $y'' = y' + 2y + 2(1-x-y^2)$

Solution.

(a) This is a homogeneous linear DE: $\underbrace{(D^2 - x)}_{L} y = \underbrace{0}_{f(x)}$

Note. This is known as the Airy equation, which we will meet again later. The general solution is of the form $C_1 y_1(x) + C_2 y_2(x)$ for two special solutions y_1, y_2. [In the literature, one usually chooses functions called $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ as y_1 and y_2. See: https://en.wikipedia.org/wiki/Airy_function]

(b) This is an inhomogeneous linear DE: $\underbrace{(x^2 D^2 + xD - (x^2+4))}_{L} y = \underbrace{x(x^2+3)}_{f(x)}$

Note. The corresponding homogeneous DE is an instance of the "modified Bessel equation" $x^2 y'' + xy' - (x^2 + \alpha^2) y = 0$, namely the case $\alpha = 2$. Because they are important for applications (but cannot be written in terms of familiar functions), people have introduced names for two special solutions of this differential equation: $I_\alpha(x)$ and $K_\alpha(x)$ (called modified Bessel functions of the first and second kind).

It follows that the general solution of the modified Bessel equation is $C_1 I_\alpha(x) + C_2 K_\alpha(x)$.

In our case. The general solution of the homogeneous DE (which is the modified Bessel equation with $\alpha = 2$) is $C_1 I_2(x) + C_2 K_2(x)$. On the other hand, we can (do it!) easily check (this is coming from nowhere at this point!) that $y_p = -x$ is a particular solution to the original inhomogeneous DE.

It follows that the general solution to the original DE is $C_1 I_2(x) + C_2 K_2(x) - x$.

(c) This is an inhomogeneous linear DE: $\underbrace{(D^2 - D - 2)}_{L} y = \underbrace{2(1-x-x^2)}_{f(x)}$

Note. We will recall in Example 17 that the corresponding homogeneous DE $(D^2 - D - 2)y = 0$ has general solution $C_1 e^{2x} + C_2 e^{-x}$. On the other hand, we can check that $y_p = x^2$ is a particular solution of the original inhomogeneous DE. (Do you recall from DE1 how to find this particular solution?)

It follows that the general solution to the original DE is $x^2 + C_1 e^{2x} + C_2 e^{-x}$.

(d) This is not a linear DE because of the term y^2. It cannot be written in the form $Ly = f(x)$.

Homogeneous linear DEs with constant coefficients

Example 17. Find the general solution to $y'' - y' - 2y = 0$.

Solution. We recall from *Differential Equations I* that e^{rx} solves this DE for the right choice of r.

Plugging e^{rx} into the DE, we get $r^2 e^{rx} - r e^{rx} - 2e^{rx} = 0$.

Equivalently, $r^2 - r - 2 = 0$. This is called the **characteristic equation**. Its solutions are $r = 2, -1$.

This means we found the two solutions $y_1 = e^{2x}$, $y_2 = e^{-x}$.

Since this a homogeneous linear DE, the general solution is $y = C_1 e^{2x} + C_2 e^{-x}$.

Solution. (operators) $y'' - y' - 2y = 0$ is equivalent to $(D^2 - D - 2)y = 0$.

Note that $D^2 - D - 2 = (D-2)(D+1)$ is the **characteristic polynomial**.

It follows that we get solutions to $(D-2)(D+1)y = 0$ from $(D-2)y = 0$ and $(D+1)y = 0$.

$(D-2)y = 0$ is solved by $y_1 = e^{2x}$, and $(D+1)y = 0$ is solved by $y_2 = e^{-x}$; as in the previous solution.