

## A crash course in linear algebra

**Example 1.** A typical  $2 \times 3$  matrix is  $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ .

It is composed of column vectors like  $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$  and row vectors like  $[1 \ 2 \ 3]$ .

Matrices (and vectors) of the same dimensions can be added and multiplied by a scalar:

For instance,  $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 5 \\ 6 & 8 & 5 \end{bmatrix}$  or  $3 \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix}$ .

**Remark.** More generally, a **vector space** is an abstraction of a collection of objects that can be added and scaled: numbers, lists of numbers (like the above row and column vectors), arrays of numbers (like the above matrices), arrows, functions, polynomials, differential operators, solutions to homogeneous linear differential equations, ...

**Example 2.** The **transpose**  $A^T$  of  $A$  is obtained by interchanging roles of rows and columns.

For instance.  $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

**Example 3.** Matrices of appropriate dimensions can also be **multiplied**.

This is based on the multiplication  $[a \ b \ c] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = ax + by + cz$  of row and column vectors.

For instance.  $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 7 & -5 \end{bmatrix}$

In general, we can multiply a  $m \times n$  matrix  $A$  with a  $n \times r$  matrix  $B$  to get a  $m \times r$  matrix  $AB$ .

Its entry in row  $i$  and column  $j$  is defined to be  $(AB)_{ij} = (\text{row } i \text{ of } A) \begin{bmatrix} \text{column } j \\ \text{of } B \end{bmatrix}$ .

**Comment.** One way to think about the multiplication  $Ax$  is that the resulting vector is a linear combination of the columns of  $A$  with coefficients from  $x$ . Similarly, we can think of  $x^T A$  as a combination of the rows of  $A$ .

Some nice properties of matrix multiplication are:

- There is an  $n \times n$  identity matrix  $I$  (all entries are zero except the diagonal ones which are 1). It satisfies  $AI = A$  and  $IA = A$ .
- The associative law  $A(BC) = (AB)C$  holds. Hence, we can write  $ABC$  without ambiguity.
- The distributive laws including  $A(B+C) = AB + AC$  hold.

**Example 4.**  $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ , so we have no commutative law.

**Example 5.**  $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

On the RHS we have the **identity matrix**, usually denoted  $I$  or  $I_2$  (since it's the  $2 \times 2$  identity matrix here).

Hence, the two matrices on the left are inverses of each other:  $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$ ,  $\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ .

The **inverse**  $A^{-1}$  of a matrix  $A$  is characterized by  $A^{-1}A = I$  and  $AA^{-1} = I$ .

**Example 6.** The following formula immediately gives us the inverse of a  $2 \times 2$  matrix (if it exists). It is worth remembering!

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{provided that } ad - bc \neq 0$$

Let's check that!  $\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} ad - bc & 0 \\ -cb + ad & ad - bc \end{bmatrix} = I_2$

In particular, a  $2 \times 2$  matrix  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  is invertible  $\iff ad - bc \neq 0$ .

Recall that this is the **determinant**:  $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$ .

$$\det(A) = 0 \iff A \text{ is not invertible}$$

**Example 7.** The system  $\begin{array}{l} 7x_1 - 2x_2 = 3 \\ 2x_1 + x_2 = 5 \end{array}$  is equivalent to  $\begin{bmatrix} 7 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ . Solve it.

**Solution.** Multiplying (from the left!) by  $\begin{bmatrix} 7 & -2 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{11} \begin{bmatrix} 1 & 2 \\ -2 & 7 \end{bmatrix}$  produces  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 1 & 2 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 13 \\ 29 \end{bmatrix}$ , which gives the solution of the original equations.

**Example 8. (homework)** Solve the system  $\begin{array}{l} x_1 + 2x_2 = 1 \\ 3x_1 + 4x_2 = -1 \end{array}$  (using a matrix inverse).

**Solution.** The equations are equivalent to  $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ .

Multiplying by  $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = -\frac{1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$  produces  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 6 \\ -4 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ .

**Example 9. (homework)** Solve the system  $\begin{array}{l} x_1 + 2x_2 = 1 \\ 3x_1 + 4x_2 = 2 \end{array}$  (using a matrix inverse).

**Solution.** The equations are equivalent to  $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ .

Multiplying by  $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = -\frac{1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$  produces  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$ .

**Comment.** In hindsight, can you see this solution by staring at the equations?

**Comment.** Note how we can reuse the matrix inverse from the previous example.

The **determinant** of  $A$ , written as  $\det(A)$  or  $|A|$ , is a number with the property that:

$$\begin{aligned} \det(A) \neq 0 &\iff A \text{ is invertible} \\ &\iff Ax = b \text{ has a (unique) solution } x \text{ for all } b \\ &\iff Ax = 0 \text{ is only solved by } x = 0 \end{aligned}$$

**Example 10.**  $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$ , which appeared in the formula for the inverse.