Homogeneous linear DEs with constant coefficients

Example 21. Find the general solution to $y'' - y' - 2y = 0$.

Solution. We recall from *Differential Equations I* that e^{rx} solves this DE for the right choice of r.
Plugging e^{rx} into the DE, we get $r^2 e^{rx} - r e^{rx} - 2e^{rx} = 0$.
Equivalently, $r^2 - r - 2 = 0$. This is called the **characteristic equation**. Its solutions are $r = 2, -1$.
This means we found the two solutions $y_1 = e^{2x}$, $y_2 = e^{-x}$.
Since this a homogeneous linear DE, the general solution is $y = C_1 e^{2x} + C_2 e^{-x}$.

Solution. (operators) $y'' - y' - 2y = 0$ is equivalent to $(D^2 - D - 2)y = 0$.
Note that $D^2 - D - 2 = (D - 2)(D + 1)$ is the **characteristic polynomial**.
It follows that we get solutions to $(D - 2)(D + 1)y = 0$ from $(D - 2)y = 0$ and $(D + 1)y = 0$.
$(D - 2)y = 0$ is solved by $y_1 = e^{2x}$, and $(D + 1)y = 0$ is solved by $y_2 = e^{-x}$; as in the previous solution.

Example 22. Solve $y'' - y' - 2y = 0$ with initial conditions $y(0) = 4$, $y'(0) = 5$.

Solution. From the previous example, we know that $y(x) = C_1 e^{2x} + C_2 e^{-x}$.
To match the initial conditions, we need to solve $C_1 + C_2 = 4$, $2C_1 - C_2 = 5$. We find $C_1 = 3$, $C_2 = 1$.
Hence the solution is $y(x) = 3e^{2x} + e^{-x}$.

Set $D = \frac{\mathrm{d}}{\mathrm{d}x}$. Every **homogeneous linear DE with constant coefficients** can be written as $p(D)y = 0$, where $p(D)$ is a polynomial in D, called the **characteristic polynomial**.

For instance. $y'' - y' - 2y = 0$ is equivalent to $Ly = 0$ with $L = D^2 - D - 2$.

Example 23. Find the general solution of $y''' + 7y'' + 14y' + 8y = 0$.

Solution. This DE is of the form $p(D)\, y = 0$ with characteristic polynomial $p(D) = D^3 + 7D^2 + 14D + 8$.
The characteristic polynomial factors as $p(D) = (D + 1)(D + 2)(D + 4)$. (Don't worry! You won't be asked to factor cubic polynomials by hand.)
Hence, by the same argument as in Example 21, we find the solutions $y_1 = e^{-x}$, $y_2 = e^{-2x}$, $y_3 = e^{-4x}$. That's enough (independent!) solutions for a third-order DE.
The general solution therefore is $y(x) = C_1\, e^{-x} + C_2\, e^{-2x} + C_3\, e^{-4x}$.

This approach applies to any homogeneous linear DE with constant coefficients!

One issue is that roots might be repeated. In that case, we are currently missing solutions. The following result provides the missing solutions.

Theorem 24. Consider the homogeneous linear DE with constant coefficients $p(D)y = 0$.

- If r is a root of the characteristic polynomial and if k is its multiplicity, then k (independent) solutions of the DE are given by $x^j e^{rx}$ for $j = 0, 1, \ldots, k - 1$.
- Combining these solutions for all roots, gives the general solution.

This is because the order of the DE equals the degree of $p(D)$, and a polynomial of degree n has (counting with multiplicity) exactly n (possibly **complex**) roots.

In the complex case. If $r = a \pm bi$ are roots of the characteristic polynomial and if k is its multiplicity, then $2k$ (independent) **real solutions** of the DE are given by $x^j e^{ax} \cos(bx)$ and $x^j e^{ax} \sin(bx)$ for $j = 0, 1, \ldots, k - 1$.

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Proof. Let r be a root of the characteristic polynomial of multiplicity k. Then $p(D) = q(D)\,(D-r)^k$.
We need to find k solutions to the simpler DE $(D-r)^k y = 0$.
It is natural to look for solutions of the form $y = c(x)e^{rx}$.
[We know that $c(x) = 1$ provides a solution. Note that this is the same idea as for variation of constants.]
Note that $(D-r)[c(x)e^{rx}] = (c'(x)e^{rx} + c(x)re^{rx}) - rc(x)e^{rx} = c'(x)e^{rx}$.
Repeating, we get $(D-r)^2[c(x)e^{rx}] = (D-r)[c'(x)e^{rx}] = c''(x)e^{rx}$ and, eventually, $(D-r)^k[c(x)e^{rx}] = c^{(k)}(x)e^{rx}$.
In particular, $(D-r)^k y = 0$ is solved by $y = c(x)e^{rx}$ if and only if $c^{(k)}(x) = 0$.
The DE $c^{(k)}(x) = 0$ is clearly solved by x^j for $j = 0, 1, \ldots, k-1$, and it follows that $x^j e^{rx}$ solves the original DE. □

Example 25. Find the general solution of $y''' = 0$.

Solution. We know from Calculus that the general solution is $y(x) = C_1 + C_2 x + C_3 x^2$.

Solution. The characteristic polynomial $p(D) = D^3$ has roots $0, 0, 0$. By Theorem 24, we have the solutions $y(x) = x^j e^{0x} = x^j$ for $j = 0, 1, 2$, so that the general solution is $y(x) = C_1 + C_2 x + C_3 x^2$.

Example 26. Find the general solution of $y''' - y'' - 5y' - 3y = 0$.

Solution. The characteristic polynomial $p(D) = D^3 - D^2 - 5D - 3 = (D-3)(D+1)^2$ has roots $3, -1, -1$.
By Theorem 24, the general solution is $y(x) = C_1 e^{3x} + (C_2 + C_3 x)e^{-x}$.

Example 27. Find the general solution of $y'' + y = 0$.

Solution. The characteristic polynomials is $p(D) = D^2 + 1 = 0$ which has no solutions over the reals.
Over the **complex numbers**, by definition, the roots are i and $-i$.
So the general solution is $y(x) = C_1 e^{ix} + C_2 e^{-ix}$.

Solution. On the other hand, we easily check that $y_1 = \cos(x)$ and $y_2 = \sin(x)$ are two solutions.
Hence, the general solution can also be written as $y(x) = D_1 \cos(x) + D_2 \sin(x)$.

Important comment. That we have these two different representations is a consequence of **Euler's identity**

$$e^{ix} = \cos(x) + i\sin(x).$$

Note that $e^{-ix} = \cos(x) - i\sin(x)$.
On the other hand, $\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$ and $\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$.
[Recall that the first formula is an instance of $\mathrm{Re}(z) = \frac{1}{2}(z + \bar{z})$ and the second of $\mathrm{Im}(z) = \frac{1}{2i}(z - \bar{z})$.]

Example 28. Find the general solution of $y'' - 4y' + 13y = 0$.

Solution. The characteristic polynomial $p(D) = D^2 - 4D + 13$ has roots $2 + 3i, 2 - 3i$.
Hence, the general solution is $y(x) = C_1 e^{2x}\cos(3x) + C_2 e^{2x}\sin(3x)$.
Note. $e^{(2+3i)x} = e^{2x}e^{3ix} = e^{2x}(\cos(3x) + i\sin(3x))$

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