Review. Every **homogeneous linear first-order DE** can be written as $y' = a(x)y$.

Its general solution is $y(x) = Ce^{\int a(x)\mathrm{d}x}$.

Comment. Note that the constant of integration can be absorbed into the factor C (in other words, there is only one degree of freedom in the above formula for the general solution).

Example 17. Solve $y' = x^2 y$.

Solution. This is a homogeneous linear first-order DE $y' = a(x)y$ with $a(x) = x^2$. Accordingly, the general solution is $y(x) = Ce^{\int a(x)\mathrm{d}x} = Ce^{\int x^2\mathrm{d}x} = Ce^{\frac{1}{3}x^3}$.

Comment. As noted above, the constant of integration gets absorbed into the factor C.

Comment. Alternatively, we can solve this DE via separation of variables (see earlier example).

Recall that, to find the general solution of the **inhomogeneous DE**

$$y' = a(x)y + f(x),$$

we only need to find a particular solution y_p.

Then the general solution is $y_p + Cy_h$, where y_h is any solution of the homogeneous DE $y' = a(x)y$.

Comment. In applications, $f(x)$ often represents an external force. As such, the inhomogeneous DE is sometimes called "driven" while the homogeneous DE would be called "undriven".

Theorem 18. (variation of constants) $y' = a(x)y + f(x)$ has the particular solution

$$y_p(x) = c(x)y_h(x) \quad \text{with } c(x) = \int \frac{f(x)}{y_h(x)}\mathrm{d}x,$$

where $y_h(x) = e^{\int a(x)\mathrm{d}x}$ is any solution to the homogeneous equation $y' = a(x)y$.

Proof. Let us plug $y_p(x) = y_h(x)\int \frac{f(x)}{y_h(x)}\mathrm{d}x$ into the DE to verify that it is a solution:

$$y_p'(x) = y_h'(x)\int \frac{f(x)}{y_h(x)}\mathrm{d}x + y_h(x)\underbrace{\frac{\mathrm{d}}{\mathrm{d}x}\int \frac{f(x)}{y_h(x)}\mathrm{d}x}_{\frac{f(x)}{y_h(x)}} = a(x)y_h(x)\int \frac{f(x)}{y_h(x)}\mathrm{d}x + f(x) = a(x)y_p(x) + f(x) \qquad \square$$

Comment. Note that the formula for $y_p(x)$ gives the general solution if we let $\int \frac{f(x)}{y_h(x)}\mathrm{d}x$ be the general antiderivative. (Think about the effect of the constant of integration!)

Recall. The formula for $y_p(x)$ can be found using **variation of constants** (sometimes called variation of parameters): that is, we look for solutions of the form $y(x) = c(x)y_h(x)$.

If we plug $y(x) = c(x)y_h(x)$ into the DE $y' = ay + f$, we find $c'y_h + cy_h' = acy_h + f$. Since $y_h' = ay_h$, this simplifies to $c'y_h = f$ or, equivalently, $c' = \frac{f}{y_h}$.

Hence, $c(x) = \int \frac{f(x)}{y_h(x)}\mathrm{d}x$, which is the formula in the theorem.

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Example 19. Solve $x^2 y' = 1 - xy + 2x$, $y(1) = 3$.

Solution. Write as $\frac{\mathrm{d}y}{\mathrm{d}x} = a(x)y + f(x)$ with $a(x) = -\frac{1}{x}$ and $f(x) = \frac{1}{x^2} + \frac{2}{x}$.

$y_h(x) = e^{\int a(x)\mathrm{d}x} = e^{-\ln x} = \frac{1}{x}$. (Why can we write $\ln x$ instead of $\ln|x|$? See comment below.) Hence:

$$y_p(x) = y_h(x)\int \frac{f(x)}{y_h(x)}\mathrm{d}x = \frac{1}{x}\int \left(\frac{1}{x} + 2\right)\mathrm{d}x = \frac{\ln x + 2x + C}{x}$$

Using $y(1) = 3$, we find $C = 1$. In summary, the solution is $y = \frac{\ln(x) + 2x + 1}{x}$.

Comment. Note that $x = 1 > 0$ in the initial condition. Because of that we know that (at least locally) our solution will have $x > 0$. Accordingly, we can use $\ln x$ instead of $\ln|x|$. (If the initial condition had been $y(-1) = 3$, then we would have $x < 0$, in which case we can use $\ln(-x)$ instead of $\ln|x|$.)

Comment. Observe how the general solution (with parameter C) is indeed obtained from any particular solution (say, $\frac{\ln x + 2x}{x}$) plus the general solution to the homogeneous equation, which is $\frac{C}{x}$.

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